Homework 2 Solutions

Math 131B-2

- (3.26) The empty set $\emptyset$ contains no points, so it is trivially true that any point in $\emptyset$ has a neighbourhood in $\emptyset$. Ergo $\emptyset$ is open. Similarly, $\emptyset$ has no limit points, so $\emptyset$ contains all its limit points and is closed. The entire space M contains all neighbourhoods of each of its points, hence is open. Furthermore, since M is the entire space it must contain all its limit points, hence M is closed.

- (3.30) Let $S \subset M$ be a finite subset of a metric space, i.e. $S = \{x_1, \cdots, x_n\}$. Let y be any point of M, and let $r = \min\{d(x_i, y) : x_i \neq y\}$. Since r is the minimum of a set of positive numbers, r is also positive, and in particular $r > 0$. Ergo $B(y; r)$ is a neighborhood of y which contains no point of S other than possibly y. Hence y is not a limit point of S. Since y was an arbitrary point of S, S has no limit points, and therefore trivially contains all its limit points. We conclude that S is closed.

- (3.31) (a) Let us show that any point $y \notin \overline{B}(a; r)$ is not a limit point of $\overline{B}(a; r)$. Given such a y, we know that $r' = d(y, a) > r$. Let $r'' = \frac{r'-r}{2}$, and consider the neighbourhood $B(y; r'')$ of y. We claim this neighbourhood contains no points of $\overline{B}(a; r)$. For, suppose there was some point x in the intersection; then we would have

  $$\begin{aligned} d(a, y) &\leq d(a, x) + d(x, y) \\ &< r'' + r \\ &= \frac{r' - r}{2} + r \\ &= \frac{r' + r}{2} \\ &< r' \end{aligned}$$

  Here the last step uses the fact that $r' > r$. But this is nonsense, since $d(a, y) = r'$. We conclude there are no points of $\overline{B}(a; r)$ in $B(y; r')$, and therefore y is not a limit point of $\overline{B}(a; r)$. Hence $\overline{B}(a; r)$ contains all its limit points and is closed.

  (b) Let (M, d) be any set with the discrete metric, and let x be a point of M. Then $B(x; 1) = \{x\}$, which has closure $\overline{B(a; r)} = \{x\}$, but $\overline{B}(x; 1) = M$.(Yes, this is sort of unhelpful notation.)

- (3.2)(a)The set S consisting of all integers has no accumulation points; every $x \in \mathbb{R}$ has a neighbourhood $B(x; r)$ which contains no integer other than possibly x. This set is closed.

(b)The set S=$(a,b]$ is neither closed nor open; the point b has no neighbourhood contained in $(a,b]$, so it isn't open, but a is a limit point not contained in the set, so it isn't closed. The set of limit points is $[a,b]$.

(c) The set $\{\frac{1}{n} : n \in \mathbb{N}\}$ is neither closed nor open; notice that $\frac{1}{n}$ has no neighbourhood contained in S, but by the same token 0 is a limit point of S not contained in S. In fact the set of limit points is exactly $\{0\}$.

(d) If $S = \mathbb{Q}$, the set of limit points of S is all of $\mathbb{R}$, since every nbhd of any point in $\mathbb{R}$ contains a rational number. But by the same token, no nbhd of a point in $\mathbb{Q}$ is contained in $\mathbb{Q}$, since every interval contains an irrational number, so S is neither closed nor open.

(f)If $S = \{(-1)^n + \frac{1}{m} : n, m \in \mathbb{N}\}$, the two limit points are $\{\pm 1\}$, and the set is neither open nor closed, for the same reasons as part (c).

(g) Let $S = \{\frac{1}{n} + \frac{1}{m} : m, n \in \mathbb{N}\}$, we see that any neighbourhood $B(\frac{1}{n}; r) = (\frac{1}{n} - r, \frac{1}{n} + r)$ must contain a point $\frac{1}{n} + \frac{1}{m})$, such that $\frac{1}{m} < r$, so $\frac{1}{n}$ is a limit point of S for all n. Moreover, every neighbourhood $B(0; r) = (-r, r)$ of 0 contains a point $\frac{1}{n} + \frac{1}{m}$ such that $\frac{1}{n}, \frac{1}{m} < \frac{r}{2}$. Therefore 0 is also a limit point. Hence S is neither open nor closed: certainly no point of S has a nbhd in S, but also S does not contain all its limit points.

- (3.12)(b)Let $S \subset T$. Let $x \in S'$ be a limit point of S. Then every ball $B(x; r)$ around x contains a point y of S other than x, and y is also a point of T. Ergo x is also a limit point of T, so $S' \subseteq T'$.

  (c) Let $x \in (S \cup T)'$, that is, let x be a limit point of $S \cup T$. We will show that x is a limit point of at least one of S and T. Suppose that x is not a limit point of S. Then there is some ball $B(x; r)$ which does not contain a point of S. Therefore for any $r' < r$, $B(x; r')$ contains no points of S. But x is a limit point of $S \cup T$, so $B(x; r') - \{x\}$ must contain a point of $S \cup T$. Ergo $B(x; r') - \{x\}$ contains a point of T, implying that every neighborhood of x contains a point of T other than x. Hence if x is not a limit point of S, x must be a limit point of T, so $x \in S' \cup T'$. This implies $(S \cup T)' \subseteq S' \cup T'$. Conversely, suppose that $x \in S' \cup T'$ is a limit point of at least one of S and T. Without loss of generality, x is a limit point of S, and every neighbourhood of x contains a point of S other than x, implying that every nbhd of x contains a point of $S \cup T$ other than x. Ergo $x \in (S \cup T)'$, so $S' \cup T' \subseteq (S \cup T)'$.

  (f) Suppose T is a closed set containing S. Then T must contain all its limit points, and in particular if x is a limit point of S, $x \in T$. Therefore $\overline{S} \subset T$. Hence $\overline{S}$ is the smallest closed subset containing T.

- (3.43) Let $x \in \text{int}A$. Then there is a nbhd $B(x;r)$ of x contained in A. In particular, $B(x;r)$ does not contain a point of $M-A$, so x is not a limit point of $M-A$, and $x \notin \overline{M-A}$. Therefore $A \subseteq M-(\overline{M-A})$. Conversely, if $x \in A \subseteq M-(\overline{M-A})$, then $x \notin M-A$ and furthermore x is not a limit point of $M-A$, implying that there is a neighbourhood $B(x;r)$ of x which contains no point of $M-A$. Therefore $B(x;r)$ is contained in A, so $x \in \text{int}A$, and $\text{int}A \subseteq M-(\overline{M-A})$. Therefore $\text{int}A = M-(\overline{M-A})$.

- (3.46)(a) First, suppose that $x \subset \int(\cap_{i=1}^n A_i)$. Then there is a neighbourhood $B(x;r)$ of x in $\cap_{i=1}^n A_i$. Then $B(x;r) \subset A_i$ for each i, and we conclude that $x \in \int A_i$. Hence $x \in \cap_{i=1}^n \int(A_i)$, and therefore $\int(\cap_{i=1}^n A_i) \subseteq \cap_{i=1}^n \int(A_i)$.

  Conversely, suppose $x \in \cap_{i=1}^n \int(A_i)$. Then $x \in \int A_i$ for each i, so there is some neighbourhood $B(x;r_i) \subset A_i$. Let $r = \min\{r_1, \cdots, r_n\}$, then $B(x;r) \subset B(x;r_i) \subset A_i$ for all $1 \leq i \leq n$, so $B(x;r) \subset \cap_{i=1}^n A_i$. We conclude that $x \in \int \cap_{i=1}^n A_i$, and therefore that $\cap_{i=1}^n \int(A_i) \subseteq \int \cap_{i=1}^n A_i$. The desired equality follows.

  Note that the important place we used finiteness above is that the minimum of $\{r_1, \cdots r_n\}$ exists and is a positive number $r > 0$.

  (b) Let F be a collection of subsets of M. Let $x \in \text{int}\bigcap_{A\in F} A$. Then there is some neighbourhood $B(x;r)$ of x contained in $\bigcap_{A\in F} A$. But then $B(x;r)$ must be contained in every set A in the intersection, so x is in $\text{int}A$ for every $A \in F$. This implies that $F \in \text{int}A$ for every A in F, so $x \in \bigcap_{A\in F} \text{int}A$. Ergo since x was arbitrary, $\text{int}\bigcap_{A\in F} A \subset x \in \bigcap_{A\in F} \text{int}A$.

  (c) Consider the sets $U_n = (-\frac{1}{n}, \frac{1}{n})$, for $n \in \mathbb{N}$ in $\mathbb{R}$ with the standard metric. Each U_n is open and equal to its own interiors, so the intersection of the interiors is $\bigcap_{i=1}^\infty U_i = \{0\}$. However, the interior of the intersection is $\text{int}\bigcap_{i=1}^\infty U_i = \text{int}\{0\} = \emptyset$. So the interior of the intersection is not equal to the intersection of the interiors.

- The interior of A is $\text{int}\,A = (.5, 1] \times (2, 2.3)$ and the closure is $\overline{A} = [.5, 1] \times [2, 2.3]$. The interior of B is empty and the closure is $\overline{B} = B$ (note the the point $(0, 2)$ is not an element of S).

- Let M be an infinite set with the discrete metric and S any infinite subset of M. Then if x is an arbitrary element of S, $S \subset B(x;2)$, so S is bounded. Moreover, if y is any point of M, the ball $B(y;\frac{1}{2})$ contains no points of S other than possibly y, so S has no limit points in M and is trivially closed. However, the collection $\{B(x;\frac{1}{2}) : x \in S\}$ is an open cover of S which is infinite (because S is infinite) and has no finite subcover (because each open set in the cover contains exactly one point of S. Ergo S is not compact.